Induced Bipartite Entanglement from Three Qubit States and Quantum Teleportation

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Only Greenberger-Horne-Zeilinger and W states are well known to have genuine tripartite entanglement in all three qubit states. The entanglement of quantum state is also well known to play an important role in various quantum information processes. Then, the following question naturally arises: which one is better between the Greenberger-Horne-Zeilinger and the W states in real quantum information processing? We try to give an answer to this question from two aspects. First, we compute the induced bipartite entanglement for a mixture consisting of Greenberger-Horne-Zeilinger and W states. If the entanglement is the only physical resource for information processing, the induced bipartite entanglement suggests that Greenberger-Horne-Zeilinger and W states are equally good. Second, we choose the bipartite teleportation scheme as an example of quantum information processing using the mixture as a quantum channel and compute the average fidelities. Our calculation shows that the W state is slightly more robust than the Greenberger-Horne-Zeilinger state when a small perturbation disturbs the teleportation process. This slight discrepancy seems to imply that entanglement is not the only resource for quantum information processing.

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I. INTRODUCTION

Recently, there has been a flurry of activity in the entanglement of the quantum states [1]. It seems to be a genuine physical resource for various forms of quantum information processing (QIP). For example, it allows a quantum computer to outperform a classical one [2]. It also plays an important role in other branches of physics. For example, entanglement may provide a promising approach to understanding the information-loss problem in black-hole physics. The entanglement between matter and gravity [3] may provide some clues for solving the various difficulties arising in black-hole physics. Therefore, understanding the general properties of the quantum entanglement in the context of quantum information theories is very important [4].

The quantum entanglement for the two-qubit states is well understood, regardless of their being pure or mixed states. For example, for any two-qubit pure states, the entanglement of the formation $\mathcal{E}$ [5] and the Groverian measure G [6], two of the basic entanglement measures, can be computed from the concurrence $\mathcal{C}$ by using the formulae

$$\mathcal{E}(\psi) = h\left(\frac{1+\sqrt{1-\mathcal{C}^2(\psi)}}{2}\right), \qquad G(\psi) = \frac{1}{\sqrt{2}}\left[1-\sqrt{1-\mathcal{C}^2(\psi)}\right]^{1/2}, \tag{1.1}$$

where $h(x) \equiv -x\log_2 x - (1-x)\log_2(1-x)$. For the two-qubit state $|\psi\rangle = \sum_{i,j=0}^{1} a_{ij}|ij\rangle$, the concurrence $\mathcal{C}(\psi)$ becomes

$$\mathcal{C}(\psi) = 2|a_{00}a_{11} - a_{01}a_{10}|. \tag{1.2}$$

Thus, it is maximal for Bell states and vanishes for factorized states. Combining Eq.(1.1) and (1.2), one can compute the entanglement of the formation and the Groverian measure for all two-qubit pure states.

The entanglement for the mixed states is, in general, defined by the convex roof construction [7, 8][1]. For example, the concurrence for the two-qubit mixed state ρ is defined as

$$\mathcal{C}(\rho) = \min \sum_i p_i \mathcal{C}(\rho_i), \tag{1.3}$$

where the minimum is taken over all possible ensembles of the pure states. The ensemble that gives the minimum value in Eq.(1.3) is called the optimal decomposition of the mixed

[1] For the Groverian measure of mixed states, there is an entanglement monotone, which does not follow the convex roof construction. See Ref. 9.

state ρ. About ten years ago, Wootters et al. [10, 11] showed how to construct optimal ensembles for arbitrary two-qubit mixed states by considering the time-reversal operation of spin-1/2 particles. Making use of these optimal decompositions, one can compute the concurrence analytically for all two-qubit states by using the simple formula

$$\mathcal{C}(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{1.4}$$

where the λ_i's are eigenvalues, in decreasing order, of the Hermitian matrix

$$\sqrt{\sqrt{\rho}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\sqrt{\rho}}.$$

Note that $\mathcal{E}(\psi)$ and $G(\psi)$ in Eq.(1.1) are monotonic functions with respect to $\mathcal{C}(\psi)$. This fact indicates that the optimal decompositions for the entanglement of the formation and the Groverian measure are the same as that for the concurrence. Thus, one can compute $\mathcal{E}$, G and $\mathcal{C}$ for all two-qubit states, regardless of their being pure or mixed.

Recently, $\mathcal{E}$, G, and $\mathcal{C}$ for the various mixed states arising in the teleportation process through noisy channels were explicitly computed [12]. Due to noises, the sender, Alice, cannot send the single-qubit state $|\psi_{in}\rangle$ to the receiver, Bob, perfectly. If Bob receives ρ_{out}, one can compute $F \equiv \langle\psi_{in}|\rho_{out}|\psi_{in}\rangle$, which measures how well the teleportation job is performed. It is shown in Ref. 12 that the mixed states entanglements $\mathcal{E}$, G, and $\mathcal{C}$ all vanish when the average of F, say $\bar{F}$, is less than 2/3, which corresponds to the best possible score when Alice and Bob communicate with each other through a classical channel [13]. This fact implies that the mixed state entanglement is a genuine physical resource for teleportation through noisy channels.

Is the entanglement the only physical resource responsible for the QIP? The purpose of this paper is to examine this question carefully. To explore this issue, we use the fact that quantum teleportation can be prefectly implemented not only by the two-qubit Bell state but also by the following Greenberger-Horne-Zeilinger (GHZ) [14] and W [15] states:

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \qquad |\psi_W\rangle = \frac{1}{2}\left(|100\rangle + |010\rangle + \sqrt{2}|001\rangle\right). \tag{1.5}$$

If we consider the three-qubit mixture defined as

$$\rho^{QC} = p|\psi_{GHZ}\rangle\langle\psi_{GHZ}| + (1-p)|\psi_W\rangle\langle\psi_W|, \tag{1.6}$$

this fact means that ρ^{QC} allows a perfect teleportation when $p = 0$ and $p = 1$. Computing the induced bipartite entanglement for ρ^{QC} and assuming that the bipartite entanglement is the only resource for the QIP, we will predict which one is better between $|\psi_{GHZ}\rangle$ and $|\psi_W\rangle$ if they are used as a channel for the real QIP. Next, we will choose the bipartite teleportation process as an example of the real QIP. Computing the average fidelities when the teleportation process is implemented using ρ^{QC} as a quantum channel, we conclude again which one is more robust in the teleportation process. If our prediction derived from the induced bipartite entanglement and our conclusion derived from the explicit calculation in the teleportation process coincide with each other, then the bipartite entanglement is the only physical resource responsible for the teleportation.

This paper is organized as follows: In Section II, we would like to briefly review the three-tangle and its optimal decomposition for the mixture ρ^{QC}. In that section, we compute three-tangle, two-tangle, and induced bipartite entanglement explicitly for ρ^{QC} that we will use in the teleportation process. The calculation of the induced bipartite entanglement enables us to understand why $|\psi_{GHZ}\rangle$ and $|\psi_W\rangle$ allow perfect quantum teleportation. It also suggests that $|\psi_{GHZ}\rangle$ and $|\psi_W\rangle$ are equally good if the entanglement is the only resource for the QIP. In Section III, we compute the average fidelity, $\bar{F}_{GHZ}$, when a mixture consisting of unperturbed GHZ state and small perturbed W state is used as a quantum channel in the teleportation process. In Section IV, we compute the average fidelity, $\bar{F}_W$, when a mixture consisting of an unperturbed W state and a small perturbed GHZ state is used as a quantum channel in the teleportation process. In Section V, we analyze the calculational results obtained in Sections III and IV and conclude that the W state is more robust than the GHZ state when a small perturbation disturbs the teleportation process. This fact implies that the bipartite entanglement is not the only physical resource for quantum information processing.

II. THE THREE-TANGLE AND COMPUTATION OF THE INDUCED BIPARTITE ENTANGLEMENT

For the three-qubit pure state ρ^{ABC}, the concurrences $\mathcal{C}_{AC}$ and $\mathcal{C}_{BC}$ for the reduced states ρ^{AC} and ρ^{BC} satisfy the following inequality [16]:

$$\mathcal{C}_{AC}^2 + \mathcal{C}_{BC}^2 \leq \mathcal{C}_{(AB)C}^2, \tag{2.1}$$

where $\mathcal{C}_{(AB)C}$ is a concurrence between the pair AB and the qubit C. The difference between the right- and left-hand sides in Eq. (2.1) is defined as a three-tangle or residual entanglement:

$$\tau_{ABC} \equiv \mathcal{C}^2_{(AB)C} - \left(\mathcal{C}^2_{AC} + \mathcal{C}^2_{BC}\right). \tag{2.2}$$

For the state $|\psi\rangle = \sum_{i,j,k=0}^{1} a_{ijk}|ijk\rangle$, τ_{ABC} becomes [16]

$$\tau_{ABC} = 4|d_1 - 2d_2 + 4d_3|, \tag{2.3}$$

where

$$\begin{aligned}
d_1 &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2, \\
d_2 &= a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} + a_{000}a_{111}a_{110}a_{001} \\
&\quad + a_{011}a_{100}a_{101}a_{010} + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001}, \\
d_3 &= a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}.
\end{aligned} \tag{2.4}$$

Thus, the generalized GHZ and W states defined as

$$|GHZ\rangle = a|000\rangle + b|111\rangle \qquad |W\rangle = c|001\rangle + d|010\rangle + f|100\rangle \tag{2.5}$$

have

$$\tau_3^{GHZ} = 4|a^2b^2| \qquad \tau_3^{W} = 0. \tag{2.6}$$

For the mixed three-qubit state ρ, the three-tangle is defined by

$$\tau_3(\rho) = \min \sum_j p_j \tau_3(\rho_j), \tag{2.7}$$

where the minimum is taken over all possible ensembles of the pure states. Thus, we should construct the optimal decomposition to compute Eq. (2.7). The construction of the optimal decompositions for arbitrary three-qubit mixed states is a highly nontrivial and formidable job and is yet unsolved. However, for the mixture of GHZ and W states given by

$$\rho(p) = p|GHZ\rangle\langle GHZ| + (1-p)|W\rangle\langle W|, \tag{2.8}$$

the optimal decomposition was explicitly constructed in Refs. 17 and 18. The expression for the three-tangle for $\rho(p)$ is

$$\tau_3(\rho(p)) = \begin{cases} 0 & \text{for } 0 \leq p \leq p_0 \\ \tau_3(p) & \text{for } p_0 \leq p \leq p_1 \\ \tau_3^{conv}(p) & \text{for } p_1 \leq p \leq 1, \end{cases} \tag{2.9}$$

where $s = 4cdf/a^2 b$ and

$$p_0 = \frac{s^{2/3}}{1+s^{2/3}}, \qquad p_1 = \max\left(p_0, \frac{1}{2} + \frac{1}{2\sqrt{1+s^2}}\right). \tag{2.10}$$

In Eq.(2.9), $\tau_3(p)$ and $\tau_3^{conv}(p)$ are given by

$$\begin{aligned} \tau_3(p) &= \tau_3^{GHZ}|p^2 - \sqrt{p(1-p)^3}s|, \\ \tau_3^{conv}(p) &= \tau_3^{GHZ}\left[\frac{p-p_1}{1-p_1} + \frac{1-p}{1-p_1}\left(p_1^2 - \sqrt{p_1(1-p_1)^3 s}\right)\right]. \end{aligned} \tag{2.11}$$

As we mentioned in the previous section, we will consider in this paper the quantum teleportation with a quantum channel ρ^{QC} defined in Eq. (1.6). Comparing Eq. (2.8) with Eq. (1.6), we have $a = b = c = 1/\sqrt{2}$ and $d = f = 1/2$, which give

$$s = 2, \quad \tau_3^{GHZ} = 1, \quad p_0 = \frac{2^{2/3}}{1+2^{2/3}} \sim 0.614, \quad p_1 = \frac{1+\sqrt{5}}{2\sqrt{5}} \sim 0.724. \tag{2.12}$$

Thus, the three-tangle for ρ^{QC} becomes

$$\tau_3(\rho^{QC}) = \begin{cases} 0 & \text{for } 0 \le p \le p_0 \\ |p^2 - 2\sqrt{p(1-p)^3}| & \text{for } p_0 \le p \le p_1 \\ \frac{1}{1-p_1}\left[(1-t_1)p - (p_1 - t_1)\right] & \text{for } p_1 \le p \le 1, \end{cases} \tag{2.13}$$

where $t_1 \equiv p_1^2 - 2\sqrt{p_1(1-p_1)^3} \sim 0.276$. Since it is simple to derive the reduced states from ρ^{QC}, one can easily compute the concurrences $\mathcal{C}_{AB}$, $\mathcal{C}_{AC}$ and $\mathcal{C}_{BC}$ by using Wootters' procedure [10, 11], whose explicit expressions are

$$\begin{aligned} \mathcal{C}_{AB} &= \begin{cases} \frac{1-p-2\sqrt{p}}{2} & \text{for } 0 \le p \le 3-2\sqrt{2} \\ 0 & \text{for } 3-2\sqrt{2} \le p \le 1, \end{cases} \\ \mathcal{C}_{AC} = \mathcal{C}_{BC} &= \begin{cases} \frac{1}{\sqrt{2}}\left[(1-p) - \sqrt{p(1+p)}\right] & \text{for } 0 \le p \le \frac{1}{3} \\ 0 & \text{for } \frac{1}{3} \le p \le 1. \end{cases} \end{aligned} \tag{2.14}$$

The induced bipartite entanglement $\mathcal{C}_{(AB)C}$ between the bipartite AB and C is defined as $4 \min \det \rho^C$, where the minimization is taken over all possible ensembles of ρ^C. It is straightforward to show that the optimal decomposition for $\mathcal{C}_{(AB)C}$ is

$$\rho^{QC} = \frac{1}{2}|Y(0)\rangle\langle Y(0)| + \frac{1}{2}|Y(\pi)\rangle\langle Y(\pi)|, \tag{2.15}$$

where

$$|Y(\theta)\rangle = \sqrt{p}|\psi_{GHZ}\rangle - \sqrt{1-p}e^{i\theta}|\psi_W\rangle; \tag{2.16}$$

thus, the induced bipartite entanglement for ρ^{QC} reduces to[2]

$$\mathcal{C}_{(AB)C} = 1 - p + p^2. \tag{2.17}$$

The p-dependence of $\mathcal{C}_{(AB)C}$ is plotted in Fig. 1. The two-tangle $\mathcal{C}_{AC}^2 + \mathcal{C}_{BC}^2$ and the three-tangle $\tau_3(\rho^{QC})$ are plotted together. It is easy to show from the figure that the Coffman-Kundu-Wootters inequality, Eq. (2.1), is satisfied in this mixture. This figure also shows that $\mathcal{C}_{(AB)C} = 1$ at $p = 0$ and $p = 1$, which indicates that pure GHZ and pure W states are maximally entangled. This fact implies that the two-party teleportation with $p = 0$ and $p = 1$ states should be perfect. This will be confirmed in the next two sections by showing that the average fidelity, $\bar{F}$, becomes 1 at these points.

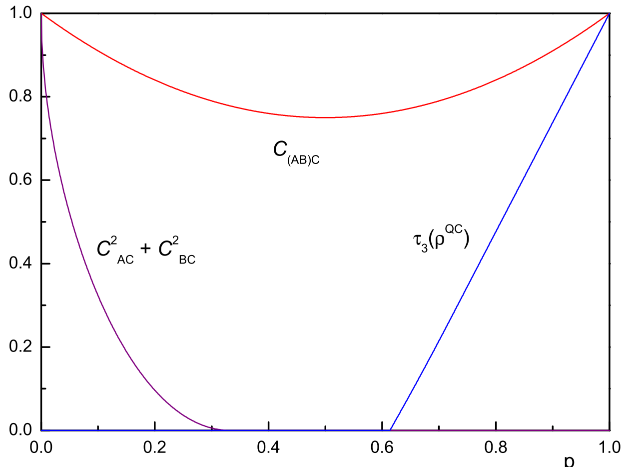

Fig. 1

FIG. 1: Plot of the p-dependence of $\mathcal{C}_{(AB)C}$, $\mathcal{C}_{AC}^2 + \mathcal{C}_{BC}^2$, and $\tau_3(\rho^{QC})$. From this figure, one can show that the Coffman-Kundu-Wootters inequality $\mathcal{C}_{(AB)C} \geq \mathcal{C}_{AC}^2 + \mathcal{C}_{BC}^2$ is satisfied. The fact that $\mathcal{C}_{(AB)C} = 1$ at $p = 0$ and $p = 1$ implies that pure GHZ and pure W states are maximally entangled.

[2] If $|\psi_{GHZ}\rangle$ and $|\psi_W\rangle$ are replaced with the generalized states $|GHZ\rangle$ and $|W\rangle$ defined by Eq. (2.5) in the mixture ρ^{QC}, the one-tangle $\mathcal{C}_{(AB)C}$ becomes

$$4 \min \det \rho^C = 4 \left[pa^2 + (1-p)(d^2 + f^2) \right] \left[pb^2 + (1-p)c^2 \right] - 4p(1-p)a^2c^2.$$

One can show easily that this reduces to $(1/9)(5p^2 - 4p + 8)$ when $a = b = 1/\sqrt{2}$ and $c = d = f = 1/\sqrt{3}$, which exactly coincides with Eq. (15) of Ref. 17.

It is worthwhile noting that the induced bipartite entanglement $\mathcal{C}_{(AB)C}$ is symmetric with respect to $p = 1/2$ line. If the entanglement is the only physical resource for the real QIP, this strongly suggests that GHZ and W states are equally good when they are used as quantum channels in the QIP. From next sections, we will choose quantum teleportation as an example of the real QIP and try to check whether or not our prediction is correct.

III. QUANTUM TELEPORTATION WITH A LARGE-p STATE

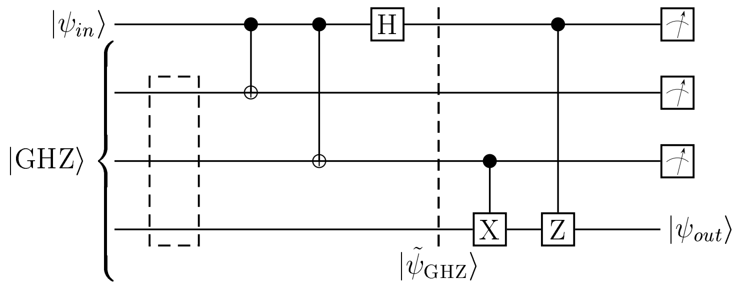

FIG. 2: Quantum circuit for quantum teleportation through noisy channels with a GHZ state. The top three lines belong to Alice while the bottom line belongs to Bob. The dotted box represents a small perturbation, which causes the quantum channel to be mixed state.

In this section, we consider the quantum teleportation with a mixed state ρ^{QC} given in Eq. (1.6) when p is large. The state ρ^{QC} with large p can be regarded as a mixed state that consists of a GHZ state plus a small perturbed W state. Therefore, we use a teleportation scheme with a GHZ state, whose quantum circuit is given in Fig. 2.

Now, we assume that the sender, Alice, who has the first two qubits in ρ^{QC}, wants to send a single qubit

$$|\psi_{in}\rangle = \cos\left(\frac{\theta}{2}\right) e^{i\phi/2}|0\rangle + \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2}|1\rangle \tag{3.1}$$

to the receiver, Bob, who has the last qubit in ρ^{QC}. Then, Fig. 2 implies that the state ρ_{out}, which Bob has finally, becomes

$$\rho_{out} = |\psi_{out}\rangle\langle\psi_{out}| = \mathrm{Tr}_{1,2,3}\left[U_{GHZ}\left(\rho_{in}\otimes\rho^{QC}\right), U^{\dagger}_{GHZ}\right] \tag{3.2}$$

where $\mathrm{Tr}_{1,2,3}$ is the partial trace over Alice's qubits and $\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$. The unitary operator U_{GHZ} can be read directly from Fig. 2, and its explicit expression is

$$U_{GHZ} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

Inserting Eq. (1.6), Eq. (3.1), and Eq. (3.3) into Eq. (3.2), one can compute ρ_{out} straightforwardly.

In order to quantify how much information is preserved or lost during this teleportation scheme, we consider a quantity

$$F(\theta, \phi) = \langle\psi_{in}|\rho_{out}|\psi_{in}\rangle, \tag{3.4}$$

which is the square of the usual fidelity defined as $F(\rho, \sigma) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$. Thus, $F = 1$ implies the perfect teleportation. For our case $F_{GHZ}(\theta, \phi)$ becomes

$$F_{GHZ}(\theta, \phi) = \frac{1}{8}\left[(3 + 5p) - (1 - p)\cos(2\theta)\right]. \tag{3.5}$$

When $p = 1$, F_{GHZ} becomes one. This means that the state $|\psi_{GHZ}\rangle$ allows perfect teleportation.

Now we define the average fidelity[3] in the form

$$\bar{F} \equiv \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta F(\theta, \phi). \tag{3.6}$$

For our case, the average fidelity $\bar{F}_{GHZ}$ becomes

$$\bar{F}_{GHZ} = \frac{5 + 7p}{12}. \tag{3.7}$$

When $p = 1$, $\bar{F}_{GHZ}$ becomes one again.

IV. QUANTUM TELEPORTATION WITH A SMALL-p STATE

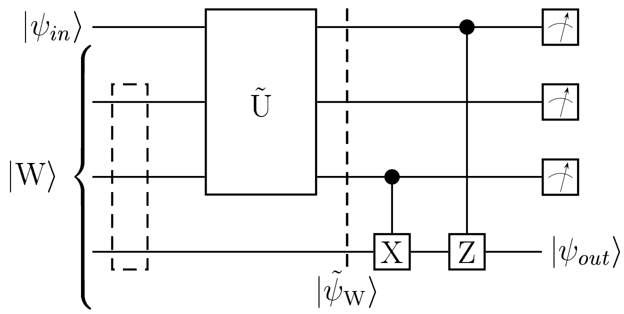

FIG. 3: Quantum circuit for quantum teleportation through noisy channels with a W state. The top three lines belong to Alice while the bottom line belongs to Bob. The dotted box represents a small perturbation, which causes the quantum channel to be mixed state. The unitary operator $\tilde{U}$ makes $|\tilde{\psi}_W\rangle$ coincide with $|\tilde{\psi}_{GHZ}\rangle$ in Fig. 2

Since the state ρ^{QC} in Eq. (1.6) with small p can be regarded as a mixed state that consists of a W state plus a small perturbed GHZ state, we use a teleportation scheme with a pure W states. The quantum circuit for this scheme is given in Fig. 3. In this figure, the unitary operator $\tilde{U}$ is introduced to make $|\tilde{\psi}_W\rangle$ be the same as $|\tilde{\psi}_{GHZ}\rangle$ in Fig. 2. The explicit expression of $\tilde{U}$ is given in Eq. (3.1) of Ref. 19.

The final state ρ_{out}, which Bob has in this teleportation process, becomes

$$\rho_{out} = |\psi_{out}\rangle\langle\psi_{out}| = \mathrm{Tr}_{1,2,3}\left[U_W\left(\rho_{in} \otimes \rho^{QC}\right)U_W^{\dagger}\right], \tag{4.1}$$

[3] Although the correct terminology is "average of the square of the fidelity," we will use "average fidelity" for simplicity throughout this paper

where $\mathrm{Tr}_{1,2,3}$ is the partial trace over Alice's qubits and $\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$. The explicit expression of U_W can be read directly from Fig. 3 in the form

$$U_W = \frac{1}{2}\begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}. \tag{4.2}$$

Inserting Eq. (1.6), Eq. (3.1), and Eq. (4.2) into Eq. (4.1), one can compute $F_W(\theta, \phi)$ and $\bar{F}_W$ straightforwardly. It is shown that $F_W(\theta, \phi)$ is independent of the angle parameters. Thus, $F_W(\theta, \phi)$ is the same as the average fidelity $\bar{F}_W$ in the form

$$\bar{F}_W = F_W(\theta, \phi) = 1 - \frac{p}{2}. \tag{4.3}$$

When $p = 0$, $\bar{F}_W$ becomes one, which indicates that the state $|\psi_W\rangle$ allows perfect teleportation.

V. DISCUSSION

We have discussed the quantum teleportation scheme using ρ^{QC} as a quantum channel. When $p = 0$ and $p = 1$, teleportation with the corresponding states is well known to be perfectly implemented. This is confirmed from the fact that the correlation between Alice and Bob, $\mathcal{C}_{(AB)C}$, becomes maximal at $p = 0$ and $p = 1$. When the teleportation scheme with a GHZ state plus small a perturbed W state is taken into account, the average fidelity of the teleportation process becomes $\bar{F}_{GHZ} = (5 + 7p)/12$. As expected, it shows a decreasing behavior with decreasing p from 1. At $p = 1/2$, where the correlation between Alice and Bob is minimized, $\bar{F}_{GHZ}$ becomes 17/24. In the teleportation scheme with a W state plus a small perturbed GHZ state, the corresponding average fidelity reduces to $\bar{F}_W = 1 - p/2$. As expected, it also exhibits a decreasing behavior with increasing p from 0. At $p = 1/2$ $\bar{F}_W$ becomes 3/4, which is slightly larger than 17/24. This fact indicates that the teleportation scheme based on a W state is more robust against small perturbations. This robustness of the W state can be concluded again from the following argument. Our result shows that $\bar{F}_W \geq \bar{F}_{GHZ}$ in the region $0 \leq p \leq p_* = 7/13 \sim 0.538$. Because $p_* > 1/2$, this means that ρ^{QC} can be regarded as a W state with a perturbed GHZ state in the wider range of p. In this aspect, we can conclude again that a W state is more robust than a GHZ state against a perturbed interaction.

It is worthwhile noting that $\bar{F}_{GHZ}$ and $\bar{F}_W$ are not symmetric to each other with respect to the $p = 1/2$ line while the induced bipartite entanglement is symmetric. This fact seems to indicate that the induced bipartite entanglement is not the only physical resource responsible for the teleportation process. Probably, the robustness for the W state is due to our choice of the bipartite teleportation scheme as an example of real QIP. We may have arrived at a different conclusion if we had chosen a different example of QIP. At least, however, we can assert that there are additional physical resources that affect real QIP. Thus, it is important to understand the additional resources to exploit various forms of QIP, such as quantum computers and quantum cryptography, in future technology. We would like to re-visit this issue in the future.

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